On Laplace Finite Marchi Fasulo Transform Of Generalized Functions

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Abstract : In this paper the Laplace Finite Marchi Fasulo Transform

$$F(n,s) = \int_{-h}^{h}\int_{0}^{\infty} f(z,t)e^{-st}\, p_n(z)dtdz$$

has been extended to certain class of generalized function and an inversion and uniqueness theorem has been derived.

Introduction : We construct an integral transform whose kernel is the product of the kernels of Laplace and Finite Marchi Fasulo transform.

The Laplace Finite Marchi Fasulo transform F (n,s) of a function f (z,t) of two variables (z , t) on the domain $\{(z,t)/-h < z < h\ ,0 < t < \infty\}$ is then defined as

$$£M\ [\ f\ (z,t)\ ] \triangleq F(n,s) = \int_{-h}^{h}\int_{0}^{\infty} f(z,t)e^{-st}\, p_n(z)dtdz \qquad \ldots\ldots(1.1)$$

We firstly extend the transform defined by (1.1) to a class of generalized function by applying similar technique used by Zemanian (4) and then we shall prove the inversion and uniqueness theorem for the class of generalized function.

Testing function spaces $£_{+,a}$ and $£_{+}(w)$:-

We shall denote the open set(-h , h) × (0, ∞) by I . Let $£_{+,a}$ denotes the space of all complex valued smooth functions φ (z, t) that are infinitely differentiable with respect to z and t on I on which the functionals λ_{a,k_1,k_2} defined by

$$\lambda_{a,k_1,k_2}(\varphi) \triangleq \sup_{\substack{0<t<\infty \\ -h<z<h}} \left| e^{at} D_t^{k_1}\, \Omega_z^{k_2}\, \varphi\,(z,t) \right|$$

assumes finite values where k_1 ,k_2 are nonnegative integers and $D_t = \frac{\partial}{\partial t}$, $D_z = \frac{\partial}{\partial z}$
$\left(\Omega_z{}^{k_2}\right)(\varphi) = (D_z^2)^{k_2}(\varphi)$

2) Now $£_{+,a}$ is linear space under the pointwise addition of functions and their multiplication

by complex numbers. Each λ_{a,k_1,k_2} is a seminorm on £+ , a and $\lambda_{a,0,0}$ is a norm, hence the countable collections $\{\lambda_{a,k_1,k_2}\}_{k_1,k_2=0}^{\infty}$ of seminorms is a countable multinorm on £+,a .

We assign to £+, a the topology generated by $\{\lambda_{a,k_1,k_2}\}_{k_1,k_2=0}^{\infty}$ making it countably multinormed space.

A sequence $\{\varphi_p\}_{p=1}^{\infty}$ converges in £+, a to φ if and only if for each pair of nonnegative integers k_1, k_2 λ_{a,k_1,k_2} $(\varphi_p - \varphi) \to 0$ as $p \to \infty$ and a sequence $\{\varphi_p\}_{p=1}^{\infty}$ is a cauchy sequence in £+ , a if and only if λ_{a,k_1,k_2} $(\varphi_p - \varphi_q) \to 0$ $\forall$ $k_1, k_2 = 0, 1, 2, \ldots$ as $p, q \to \infty$

Lemma (1.1) :- The space £+ , a is complete and therefore it is a Frechet space.

Proof : Let $\{\varphi_p\}_{p=1}^{\infty}$ be a cauchy sequence in £+ , a . Then there exists a smooth function φ (z , t) such that for each pair of non negative integers k_1 , k_2 $D_t^{k_1}$ $\Omega_z^{k_2}$ $\varphi_p(z,t) \to$ $D_t^{k_1}$ $\Omega_z^{k_2}$ $\varphi_p(z,t)$ as $p \to \infty$. Moreover given any $\in > 0$ there exists N_{k_1,k_2} such that for every $p, q > N_{k_1,k_2}$ $| e^{at} D_t^{k_1} \Omega_z^{k_2} [\varphi_p(z,t) - \varphi_q(z,t)] | < \in$, for all z .

Taking the limit as $q \to \infty$, we obtain

$| e^{at} D_t^{k_1} \Omega_z^{k_2} [\varphi_p(z,t) - \varphi(z,t)] | < \in \quad 0 < t < \infty, -h < z < h$ ……….(1.2)

Thus as $p \to \infty$ λ_{a,k_1,k_2} $\big(\varphi_p(z,t) - \varphi(z,t)\big) \to$ 0 for each k_1 , k_2 .

Finally because of the uniformity of the convergence and the fact that each $e^{at} D_t^{k_1} \Omega_z^{k_2} [\varphi_p(z,t)]$ is bounded on $0 < t < \infty, -h < z < h$ there exists constant C_{k_1,k_2} not depending on p such that $| e^{at} D_t^{k_1} \Omega_z^{k_2} [\varphi_p(z,t)] | < C_{k_1,k_2}$ for all z , t .

Therefore (1.2) implies that $| e^{at} D_t^{k_1} \Omega_z^{k_2} [\varphi_p(z,t)] | < C_{k_1,k_2} + \epsilon$ which shows that the limit function φ is a member of £+,a hence £+,a is complete , so it is Frechet space .

The countable union space £+ (w) :-

Let w denote either a finite real number or $-\infty$. We choose the monotonic sequence

Since the convergence is uniform and $\{a_p\}_{p=1}^{\infty}$ such that $a_p \to w_+$.

Then $\left\{ £_{+,a_p} \right\}_{p=1}^{\infty}$ is a sequence of testing function spaces such that $£_{+,a_p} \subset £_{+,a_{p+1}}$ for all p and the topology of $£_{+,a_p}$ is stronger than the topology induced on it by $£_{+,\,a_{p+1}}$

$$Define \qquad £_{+}(w) = \bigcup_{p=1}^{\infty} £_{+,\,a_p} .$$

A sequence $\{\varphi_p\}_{p=1}^{\infty}$ converges in $£_+$ (w) it converges in $£_{+,a_p}$ for some p . Then with these properties $£_+$ (w) is countable union space.

$£_+$ (w) is a linear space. Further it is complete space, since for fixed p $£_{+,a_p}$ is complete.

The dual spaces $£'_{+,a}$ and $£'_{+}$(w) :-

The dual space $£'_{+,a}$ of $£_{+,a}$ is the collection of all continuous linear functionals on $£_{+,a}$.

Since $£_{+,a}$ is complete , $£'_{+,a}$ is also complete.

If $a \le c$, then $£_{+,c} \subset £_{+,a}$ and the topology of $£_{+,c}$ is stronger than the topology induced on it by $£_{+,a}$. Therefore the restriction of any member $f \in £'_{+,a}$ to $£_{+,c}$ is in $£'_{+,c}$.

We assign the customary weak topology generated by the multinorm $\{ \xi_\varphi \}$, $\varphi \in £_{+,a}$ to the dual space $£'_{+,a}$, where $\xi_\varphi(f) = |\langle f , \varphi \rangle| \quad \varphi \in £_{+,a}$.

We denote by $£'_+$ (w) the dual space of $£_+$ (w) . since all $£_{+,a_p}$ are complete and $£_+$ (w) is complete. the dual space $£'_+$ (w) is also complete.

The properties of testing function spaces and their duals :-

1) $D \subset £_{+,a}$ and the convergence in D (I) implies the convergence in $£_{+,a}$ (I). Therefore the restriction of any member of $£'_{+,a}$ (I) to D(I) is in D'(I).

Similarly D(I) is subspace of $£_+$(w) whatever may be the value of w. The convergence in D(I) implies the convergence in $£_+$(w) and the restriction of any member of $£'_+$(w) to D(I) is a member of D'(I).

The member of $£'_{+,a}$ (I) and $£'_+$(w) are called distribution.

2) D(I) is dense in $£_+$ (w) for every w. Therefore $£'_+$(w) is subspace of D'(I).

3) For each $f \in £'_{+,a}$ (I) there exists a +ve constant c and a nonnegative integer r , such that for all $\varphi \in £_{+,a}$ (I) $\quad |\langle f , \varphi \rangle| \le c \max \lambda_{a,k_1,k_2}(\varphi)$

$$0 \le k_1 \le r$$

$$0 \le k_2 \le r$$

4) If f (z,t) is a locally integrable function defined on the interval I such that

$$\int_{-h}^{h}\int_{0}^{\infty} |e^{-at} f(z,t)\ dtdz\ | \quad \text{exists then f (z , t)generates a reguler member of } £_{+,\ a}(I)$$

through the definition

$$< f , \varphi > = \int_{-h}^{h}\int_{0}^{\infty} f(z,t)\ \varphi\ (z,t) dtdz \qquad \varphi \in £_{+,\ a}(I) \quad \ldots\ldots (1.3)$$

$$| < f , \varphi > | = | \int_{-h}^{h}\int_{0}^{\infty} \frac{f\ (z\ , t\ )}{e^{at}}\ e^{at}\ \varphi\ (z,t)\ dtdz\ |$$

$$\leq \quad \lambda_{a\ ,0,0} \int_{-h}^{h} \int_{0}^{\infty} |\ e^{-at}\ f(z,t)\ \ |dtdz$$

which exists in view of our assumption .

Therefore (1.3) defines a functional f on $£_{+,\ a}$ (I) , this functional is linear. Further if $\{\varphi_p\}_{p=1}^{\infty}$ converges in $£_{+,\ a}$ (I) to zero then $\lambda_{a,0,\ 0}$ (φ_p) →0 so that

|< f, φ_p >| → 0 therefore f is continuous on $£_{+,\ a}$ (I) .

Similarly if w < a , then f generates a regular distribution of $£'_{+}$ (w) through the distribution

$$< f , \varphi > = \int_{-h}^{h}\int_{0}^{\infty} f(z,t)\ \varphi\ (z,t) dtdz \qquad \varphi \in £_{+,\ a}(I)\ ,$$

$$where \int_{-h}^{h}\int_{0}^{\infty} |\ e^{-at}\ f(z,t)\ \ |dtdz \quad exists\ .$$

This condition is satisfied if $e^{-\sigma t}$ f (z ,t) bounded on o < t < ∞ for every choice of σ where σ>w.

5) For each positive integer n and Re. S ≥ a the function $e^{-st}\ p_n(z)$ is a member of $£_{+,\ a}$ (I) and is also member of $£_{+}$ (w) if a > w .

Proof :- e^{at} [$D_t^{k_1}\ \Omega_z^{k_2}\ e^{-st}\ p_n(z)$]

$$= \ (-1)^{k_1+k_2} \quad s^{k_1}\ (a_n^2)^{k_2}\ e^{-(s-a)t}\ p_n(z) \ \text{ for } t \geq 0 \ \ -h < z < h\ .$$

The right hand side of above equation is bounded for Re . s $\geq$ a for the positive eigen values a_n.

Thus for each k_1 , k_2 , λ_{a,k_1,k_2} [e^{-st} $p_n(z)$] exists which shows that e^{-st} $p_n(z)$ belongs to $£_{+,\,a}$ (I) In the similer way we can show that for every a > w , e^{-st} $p_n(z)$ ϵ $£_+$(w) .

The Generalized Laplace Finite Marchi Fasulo Transform :-

We call the generalized function f as Laplace Finite Marchi Fasulo transformable if it belongs to $£'_+$(w) for some real number w.

Let σ_f be defined as , σ_f = inf $\{$ w | f ϵ $£'_+$(w) $\}$

If f is Laplace Finite Marchi Fasulo transformable function , then we see that

∃ σ_f s . t . f ϵ $£'_+$(w) ∀ w < σ_f Thus for a given Laplace Finite Marchi Fasulo transformable function f ϵ $£'_+$(w) if D_f denote strip of definition i .e. D_f = $\{$ (n ,s) / σ_f < Re s ,n +ve integer $\}$ .Then Laplace Finite Marchi Fasulo transformation F (n, s) of f (z, t) is defined by

£M [f (z, t)] ≜ F (n, s) = < f (z, t) , $e^{-st}\, p_n(z)$ >

i.e , it is defined as the application of f ϵ $£'_+$ (σ_f) to kernel $e^{-st}\, p_n(z)$ ϵ $£_+$ (σ_f) or equivalently

f ϵ $£'_{+,\,a}$ to $e^{-st}\, p_n(z)$ ϵ $£_{+,a}$ for any σ_f < a $\leq$ Re. S < f (z , t) , e^{-st} $p_n(z)$ >

Boundedless property of Generalised Laplace Finite Marchi Fasulo Transform:

We show that the generalized Laplace Finite Marchi Fasulo transform F (n, s) defined as above is bounded for (n, s) ϵ D_f .

Theorem : - Let f ϵ $£'_{+,\,a}$ (I) and F (n, s) = < f (z, t) , $e^{-st}\, p_n(z)$ >

For (n, s) ϵ D_f . Then F (n, s) satisfies the inequality

| F (n, s) | $\leq$ C . A .

Proof : - Let f ϵ $£'_{+,\,a}$ (I) . Then by property (3) above there exists a non negative integer r and a positive constant C such that for Re. S ≥ a

$$| F (n, s) | = | < f (z, t) , e^{-st}\, p_n(z) > |$$

$$\leq C \max_{\substack{0 \leq k_1 \leq r \\ 0 \leq k_2 \leq r}} \lambda_{a,k_1,k_2} [e^{-st}\, p_n(z)]$$

$$\leq C \max_{\substack{0 \leq k_1 \leq r \\ 0 \leq k_2 \leq r}} \sup_{(z,t) \in I} | e^{at} D_t^{k_1} (D_z^2)^{k_2} e^{-st} p_n(z) |$$

$$= C \max_{\substack{0 \leq k_1 \leq r \\ 0 \leq k_2 \leq r}} \sup_{(z,t) \in I} | e^{at} s^{k_1} a_n^{2k_2} e^{-st} p_n(z) |$$

but $| e^{at} e^{-st} p_n(z) | < A$ on $0 < t < \infty$, $-h < z < h$

so that

$$|F(n,s)| \leq C . A \max_{\substack{0 \leq k_1 \leq r \\ 0 \leq k_2 \leq r}} | s^{k_1} . a_n^{2k_2} |$$

$$|F(n, s)| \leq C . A . \quad P\left(|S| \ (a_n^2)^r\right)$$

where $P\left(|S| \ (a_n^2)^r\right)$ is alpolynomial that depends in general on the choice of a .

Analyticity Theorem : -

Theorem : -The Generalized Laplace Finite Marchi Fasulo transform is an analytic function of s
proof : Let (n, s) be arbitrary but fixed point in D_f and choose the real number a and r such that

$$\sigma_f < a \leq \text{Re. } S - r \leq \text{Re. } S + r$$

let Δs be complex increment such that $|\Delta s| < r$.

Now for $\Delta s \neq 0$ by definition of F (n, s) as

$$\frac{F(n, s + \Delta s) - F(n, s)}{\Delta s} - \left< f(z, t), \frac{\partial}{\partial s} e^{-st} p_n(z) \right>$$

$$= \left< f(z, t), \psi_{\Delta s}(z, t) \right. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.4)$$

where $\psi_{\Delta s}(z,t) = \{ \frac{1}{\Delta s}[e^{-(s+\Delta s)t} - e^{-st}] - \frac{\partial}{\partial s} e^{-st} \} p_n(z)$ …………………(1.5)

since $\psi_{\Delta s}(z,t) \in £_{+,a}(I)$ equation (1) has meaning .

We show that $\psi_{\Delta s}(z,t) \to 0$ in $£_{+,a}(I)$ as $\Delta s \to 0$ but as $f \in £'_{+,a}(I)$ this implies that

$< f , \psi_{\Delta s} > \to 0$

let c denote the circle with center at s and radious r_1 where $o < r < r_1 <$ Re. S – a

by equation (1.4)

$$D_t^{k_1}\,\Omega_z^{k_2}\ \psi_{\Delta s}(z,t) = (-1)^{k_1+k_2}\left(a_n^{\,2}\right)^{k_2} p_n(z)\ \{ \frac{1}{\Delta s}[((s+\Delta s)^{k_1} e^{-(s+\Delta s)t} - s^{k_1}e^{-st}] - s^{k_1} e^{-st}\}$$

since e^{-st} $p_n(z)$ is analytic in s using Cauchy's integral formula to the r.h.s. of the last equation we obtain

$$D_t^{k_1}\,\Omega_z^{k_2}\ \psi_{\Delta s}(z,t) = (-1)^{k_1+k_2}(a_n^{\,2})^{k_2}\, p_n(z)$$

$$\left[\frac{1}{2\pi i}\,\frac{1}{\Delta s}\int_c \frac{\xi^{k_1}\ e^{-\xi t}\,d\xi}{\xi-(s+\Delta s)} - \int_c \frac{\xi^{k_1}\ e^{-\xi t}\,d\xi}{\xi - s} - \frac{1}{2\pi i}\int_c \frac{\xi^{k_1}\ e^{-\xi t}\,d\xi}{(\xi-s)^2}\right.$$

$$= (-1)^{k_1+k_2}(a_n^{\,2})^{k_2}\, p_n(z)\ \times$$

$$\frac{1}{2\pi i}\int_c \frac{\xi^{k_1}\ e^{-\xi t}\,d\xi}{(\xi-s)(\xi-s-\Delta s)} - \int_c \frac{\xi^{k_1}\ e^{-\xi t}\,d\xi}{(\xi-s)^2}$$

$$= (-1)^{k_1+k_2}\left(a_n^{\,2}\right)^{k_2} p_n(z)\,\frac{\Delta s}{2\pi i}\int_c \frac{\xi^{k_1}\ e^{-\xi t}\,d\xi}{(\xi-s-\Delta s)(\xi-s)^2}$$

now for all $\xi \in C$ and $0 < t < \infty$, $-h < z < h$

$$\left|e^{at} D_t^{k_1}\,\psi_{\Delta s}\,\Omega_z^{k_2}\ (z,t)\right| = (a_n^{\,2})^{k_2}\ |p_n(z)|\ \frac{|\Delta s|}{2\pi}\int_c \frac{|e^{at}\ \xi^{k_1} e^{-\xi t}\,||d\xi\,|}{(\xi-s-\Delta s)(\xi-s)^2}$$

$$\le \frac{|\Delta s|}{2\pi}.\,k\,(a_n^{\,2})^{k_2}\int_c \frac{|\,d\xi\,|}{((r_1-r)r_1^{\,2}}$$

$$\le \frac{|\,\Delta s\,|\,k'(a_n^{\,2})^{k_2}}{(r_1-r)\,r_1}\ ,$$

where $|\,e^{at}\ \xi^{k_1}\ e^{-\xi t}\ |\ |\ p_n(z)]\ | \le k$ k being constant independent of ξ and t .

The right hand side of the last equality is independent of z ,t converges to zero as $|\,\Delta s\,| \to o$

this proves that $\Psi_{\Delta s}$ (z ,t) converges to zero in $£_{+,\,a}$ (I) as $\Delta s \to$ o.

Hence if f ∈ $£'_{+,\,a}$ (I) , < f , $\Psi_{\Delta s}$ > → o as $\Delta s \to$ o.

Lemma (1.2) : - Let £M [f (z ,t)] = F (n ,s) ,

where (n ,s) ∈ D_f , Re. S > σ and φ (z ,t) ∈ $£_{+}$,a (I) and -h < a_1 < b_1 < h

assume that ,

$$\varphi\,(n\,,s)\;=\int_{a_1}^{b_1}\int_{0}^{\infty}\;\varphi(z,t)\;e^{st}\;\frac{p_n(z)}{\lambda_n}\;dt\,,dz\,.$$

Then for any fixed real number r with o < r < ∞

$$\int_{-r}^{r} < f(u\,,v),\; e^{-su}\,p_n(v) > \varphi\,(\,n\,,s)d\rho$$

$$=\;<\;f(u\,,v),\int_{-r}^{r}\;e^{-su}\,p_n(v)\;\;\varphi\,(\,n\,,s)d\rho\;\;>,$$

where s = σ + iρ and σ is fixed and σ > σ_f .

Proof : - We know that

e^{-st} $p_n(z)$ is a member of $£_{+}$,a (I) and

$\int_{-r}^{r}$ $e^{-su}\,p_n(v)\,\varphi\,(\,n\,,s)\;d\rho$ is also the member of £+ , a (I)

is also the member of $£_{+}$,a (I

indeed ,

$$D_u^{k_1}\;D_v^{2k_2}\int_{-r}^{r}\;e^{-su}\,p_n(v)\,\varphi\,(\,n\,,s)\;d\rho$$

$$\le\;e^{-\sigma u}\int_{-r}^{r} s^{k_1}\;(a_n{}^2)^{k_2}|\,p_n(v)|\;\;|\,\varphi\,(\,n\,,s)\,|d\rho$$

≤ A . $e^{-\sigma u}$ s^{k_1} $(a_n{}^2)^{k_2}$. 2r which exists , where A is bound for φ (n ,s) .
$p_n(v)$ both sides has sence .

If φ (z ,t) = o , the proof is obvious. So assume that φ (z ,t) ≠ o .

partition the path of integration on the straight line s = σ – ir to s = σ + ir into m intervals each of length $\frac{2r}{m}$ and let $s_p = \sigma + i\,\rho_p$ be any point in path interval

$$\theta_m(u,v) = \sum_{p=1}^{m} e^{-s_p u}\; p_n(v)\;\varphi\,(n\,, s_p)\;\frac{2r}{m} \qquad \ldots\ldots\ldots..(1.6)$$

by applying f (u ,v) to (1.5) term by term

$$< f(u,v), \theta_m(u,v) >= \sum_{p=1}^{m} < f(u,v), e^{-s_p u}\; p_n(v) > \;\varphi\,(n\,, s_p)\;\frac{2r}{m}$$

$$\to \int_{-r}^{r} < \; f\;(u\,,v)\, e^{-su}\, p_n(v)\;\;\varphi(\,n\,,s)\, d\rho.$$

Since < f (u ,v) , e^{-su} $p_n(v)$ > φ (n ,s) is a continuous function of ρ

choose a such that $\sigma_f < a < \sigma$ since f ∈ $£_{+}$,a

To show that

$$\textit{To show that}\;\; \theta_m(u\,,v) \text{ converges in } £_{+}\,,a\,(I)\; to \int_{-r}^{r} e^{-su}\, p_n(v)\,\varphi(\,n\,,s)\, d\rho\,,$$

We will to show that for fixed k_1 , k_2

A_m (u ,v) convsrges uniformly to zero , o < u < ∞ , -h < v < h , where

$$A_m(u\,,v) \;=\; e^{au}\, D_u{}^{k_1}\;\; \Omega_v{}^{k_2}\,[\,\theta_m(\,u\,,v) - \int_{-r}^{r} e^{-su}\, p_n(v)\,\varphi\,(\,n\,,s)\, d\rho\,]\,.$$

$$A_m(u\,,v) \;=\; e^{au}\,(-1)^{k_1+\,k_2} \sum_{p=1}^{m} s_p^{k_1}\;\;(a_n{}^2)^{k_2}\, e^{-s_p u}\;\; p_n(v)\;\;\varphi(\,n\,,s)\;\frac{2r}{m} \;-$$

$$(-1)^{k_1+\,k_2} \int_{-r}^{r} s^{k_1}\;\;(a_n{}^2)^{k_2}\;\; e^{-su}\, p_n(v)\,\varphi(\,n\,,s)\, d\rho\,,$$

Now | e^{au} . e^{-su} $p_n(v)$| ≤ $e^{(a-\sigma)\,u}$. $p_n(v)$→ o as |u| → ∞ as a < σ .

So given ∈ > 0 we choose T so large that for all |u| > T

$$|e^{au}.\ e^{-su}\, p_n(v) \leq \frac{\epsilon}{2}\ [\int_{-r}^{r} |\, s^{k_1}\ (a_n{}^2)^{k_2}\ \varphi\ (\,n\,,s)|\ d\rho\ ]^{-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.7)$$

since φ (z ,t) ≠ o the right hand side of (1.7) is finite

Hence for all |u| > T the magnitude of the second term on the right hand side of (1.6) is bounded by $\frac{\epsilon}{2}$.

Morever for |u| > T the magnitude of first term on right hand side of (1.6) is bounded by

$$\frac{\epsilon}{2}\ [\int_{-r}^{r} |\, s^{k_1}\ (a_n{}^2)^{k_2}\ \varphi\ (\,n\,,s)|d\rho\ ]^{-1} \sum_{p=1}^{m}\ s^{k_1}\,(a_n{}^2)^{k_2}\,\varphi\big(\,n\,,s_p\big)\ \frac{2r}{m}.$$

We can now Choose m_0 so large that for all $m > m_0$ the last expression is less than ϵ .

Hence for all |u| > T and for all $m > m_0$ | $A_m(u\,,v)$ | < ϵ .

* Theorem (Inversion) :-

Let f ϵ $£_{+,\,a}$ (I) , and let F (n ,s) be the distributional Laplace Finite Marchi Fasulo transform of f (z ,t) .

defined by

£M [f (z, t)] ≜ F (n, s) = < f (z, t) , $e^{-st}\, p_n(z)$] >

For (n ,s) ϵ D_f , in the sense of convergence in D' (I) ,

$$f(z,t) = \lim_{r,m\to\infty} \left[\ \frac{1}{2\pi i}\ \sum_{n=1}^{m} \frac{p_n(z)}{\lambda_n} \int_{\sigma-ir}^{\sigma+ir} F(n\,,s)\, e^{st}\, ds\ \right],$$

where σ is any fixed number s .t. $\sigma > \sigma_f$.

Proof : - Let φ (z ,t) be an arbitray member of D (I) . To show that

$$\lim_{r,m\to\infty} < \frac{1}{2\pi i}\ \sum_{n=1}^{m} \frac{p_n(z)}{\lambda_n} \int_{\sigma-ir}^{\sigma+ir} F(n\,,s)\, e^{st}\, ds\ \ \varphi(z\,,t) > \ \to < f\ (u,v)\,,\varphi\ (u,v) > .$$

…………..(1.8)

Let φ ϵ D (I) has support contained in [A , B] × [a_1 , b_1] ,

where $0 < A < B < \infty$ and $-h < a_1 < b_1 < h$.

$$\frac{1}{2\pi i} \sum_{n=1}^{m} \frac{p_n(z)}{\lambda_n} \int_{\sigma-ir}^{\sigma+ir} F(n\,,s)\, e^{st}\, ds \text{ is locally } integrable\ function\ on\ I.$$

Equation (1.7) can be written without limit notation as

$$\frac{1}{2\pi i} \int_{a_1}^{b_1} \int_{0}^{\infty} [\ \sum_{n=1}^{m} \frac{p_n(z)}{\lambda_n} \int_{\sigma-ir}^{\sigma+ir} F(n\,,s)\, e^{st}\, ds\ ]\ \varphi(z\,,t)\ dt\, dz$$

substituting s = $\sigma + i\rho$ ds = idρ we get

$$\frac{1}{2\pi} \int_{a_1}^{b_1} \int_{0}^{\infty} [\ \sum_{n=1}^{m} \frac{p_n(z)}{\lambda_n} \int_{-r}^{r} F(n\,,s)\, e^{st}\, d\rho\ ]\ \varphi(z\,,t)\ dt\, dz\,.$$

We interchange the order of integration as φ (z ,t) has bounded support and integrand is continuous function of z , t and ρ.

Therefore last expression takes the form

$$\frac{1}{2\pi} \sum_{n=1}^{m} \int_{-r}^{r} F(n\,,s) \int_{a_1}^{b_1} \int_{0}^{\infty} \varphi(z\,,t)\ e^{st}\ \frac{p_n(z)}{\lambda_n}\ dt\, dz\ d\rho$$

$$= \frac{1}{2\pi} \sum_{n=1}^{m} \int_{-r}^{r} <\ f(u,v)\, e^{-su}\ p_n(v) > \int_{a_1}^{b_1} \int_{0}^{\infty} \varphi(z\,,t)\ e^{st}\ \frac{p_n(z)}{\lambda_n}\ dt\, dz\ d\rho$$

$$= <f\ (u\,,v), \frac{1}{2\pi} \sum_{n=1}^{m} \int_{-r}^{r} e^{-su}\ p_n(v) \int_{a_1}^{b_1} \int_{0}^{\infty} \varphi(z\,,t)\ e^{st}\ \frac{p_n(z)}{\lambda_n}\ dt\, dz\ d\rho >$$

the order of integration for repeated integrals can be changed because again φ(z , t) is of bounded support and the integrand is continuous function of (t ,z , ρ), we obtain

$$<f\ (u\,,v), \frac{1}{2\pi}\ [\ \sum_{n=1}^{m} \int_{-r}^{r} e^{-su}\ p_n(v) \int_{a_1}^{b_1} \int_{0}^{\infty} \varphi(z\,,t)\ e^{st}\ \frac{p_n(z)}{\lambda_n}\ dt\, dz\ d\rho >$$

$$= <f\ (u\,,v), \frac{1}{2\pi} \sum_{n=1}^{m} \int_{a_1}^{b_1} \int_{0}^{\infty} \varphi(z\,,t)\ \frac{p_n(z)\ p_n(v)}{\lambda_n}\, dt\, dz \int_{-r}^{r} e^{\sigma(t-u)}\ e^{(t-u)i\rho} d\rho\ >$$

$$= < f\,(u\,,v), \frac{1}{2\pi} \sum_{n=1}^{m} \int_{a_1}^{b_1} \int_{0}^{\infty} \varphi(\mathrm{z}\,,\mathrm{t})\ \mathrm{T_n}\ \ e^{\,\sigma(t-u)}\ dt\,dz\ \frac{e^{i(t-u)r} - e^{-i(t-u)r}}{i(t-u)} >,$$

$$\text{where}\quad \mathrm{T_n} = \sum_{n=1}^{m} \frac{p_n\,(z)\ \ p_n\,(v)}{\lambda_n}$$

$$= < f\,(u\,,v), \quad \frac{1}{\pi} \int_{a_1}^{b_1} \int_{0}^{\infty} \varphi(\mathrm{z}\,,\mathrm{t})\ \mathrm{T_n}\ \ e^{\,\sigma(t-u)}\ \frac{\sin r\,(t-u)}{(t-u)}\ dtdz >$$

$\rightarrow$ < f (u ,v) , φ (u , v) > as r , m $\rightarrow \infty$.

The Uniqueness Theorem : -

Let £M [f (z ,t)] = F (n ,s) and

£M [g (z ,t)] = G (n ,s)

for all (n ,s) Є D_f and (n ,s) Є D_g respectively and if

F (n ,s) = G (n ,s) for (n ,s) Є $D_f \wedge D_g \neq 0$,

then f = g in sense of equality in D' (I) .

Proof :- In the sense of convergence in D' (I) and in view of inversion theorem

$$f(z,t) = \lim_{r,m\to\infty} \frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{p_n(z)}{\lambda_n} \int_{\sigma-ir}^{\sigma+ir} F(n\,,s)\,e^{st}\,ds\,,$$

the right hand side of this equation becomes

$$\lim_{r,m\to\infty} \frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{p_n(z)}{\lambda_n} \int_{\sigma-ir}^{\sigma+ir} G(n\,,s)\,e^{st}\,ds$$

which by inversion theorem equal to g (z,t)

Hence f = g .

Lemma 3 : -

if ψ (z ,t) Є D (I) then

$$\frac{1}{\pi}\, e^{au} \int_{a_1}^{b_1}\int_{0}^{\infty} \quad T_n \;\frac{\sin r(t-u)}{t-u}\;\left[e^{\sigma(t-u)}\,\psi\,(z\,,t) - \;\psi\,(u\,,v)\,\right] dtdz$$

converges to zero uniformly an $0 < u < \infty$, as $r, m \to \infty$

where the support of $\psi\,(z\,,t)$ is contained in $[A, B]\,[a_1, b_1]$

$O < A < B < \infty$, $-h < a_1 < b_1 < h$

Proof : - Let us devide the interval $(0, \infty)\,(-h, h)$ in to six disjoint sets .

(1) $(0, A)\;U\,(B, \infty) \times (-h, h)$

(2) $(A, B) \times (b_1, h)$

(3) $(A, B) \times (-h, a_1)$

(4) $(0, A) \times (a_1, b_1)$

(5) $(B, \infty) \times (a_1, b_1)$

(6) $(A, B) \times (a_1, b_1)$

Case I : -

Proof : - for $(u, v) \in (o, A)\;U\,(B, \infty) \times (-h, h)$

$\psi\,(u\,,v) = 0$ since $\psi\,(u\,,v)$ has support contained in $[A, B] \times [a_1, b_1]$

therefore

$$\frac{e^{au}}{\pi} \int_{a_1}^{b_1}\int_{0}^{\infty} \quad T_n \;\frac{\sin r(t-u)}{t-u}\; e^{\sigma(t-u)}\,\psi\,(z\,,t) - \;\psi\,(u\,,v)\; dtdz$$

$$= \frac{e^{au}}{\pi} \int_{a_1}^{b_1}\int_{0}^{\infty} \quad T_n \;\frac{\sin r(t-u)}{t-u}\; e^{\sigma(t-u)}\,\psi\,(z\,,t)\; dtdz$$

By lemma 2 as $r \to \infty$ this integral reduces to

$$= \frac{e^{au}}{\pi} \int_{a_1}^{b_1} \quad T_n \quad \psi\,(z\,,v)\; dz$$

we shall now show that for fixed u and $-h < z < h$

$$\lim_{m \to \infty} \frac{e^{au}}{\pi} \int_{a_1}^{b_1} \mathrm{T_n} \quad \psi\,(z\,,v)\, dz \quad converges\ to\ zero$$

uniformly for all z . since $\psi\,(z\,,v) \in D\,(I_u)$

where I_u { (z, v) / -h< z < h ,u is fixed }

$$since\ e^{\,au} \int_{a_1}^{b_1} \mathrm{T_n}\ \psi\,(z\,,v)\, dz \quad bounded\ on \quad 0 < v < \infty\ the\ last\ integral$$

converges to zero Uniformly by lemma 2 as m→∞ in the same way we prove the remaining cases .

Case 6:-

proof we show that the given equation is also true for the interval $[\,A\,, B\,] \times [\,a_1\,,\ b_1]$

the given equation can be written as ,

$$\frac{e^{au}}{\pi} \int_{a_1}^{b_1} \int_{0}^{\infty} \mathrm{T_n} \ \frac{\sin r(t-u)}{t-u} \ e^{\sigma(t-u)}\ \psi\,(z\,,t)\, dtdz$$

$$-\frac{e^{au}}{\pi} \int_{a_1}^{b_1} \int_{0}^{\infty} \mathrm{T_n} \ \frac{\sin r(t-u)}{t-u} \quad \psi\,(u\,,v)\, dtdz$$

this tends to

$$e^{au} \int_{a_1}^{b_1} \mathrm{T_n} \ \ \psi\,(z\,,u) dz - e^{au} \int_{a_1}^{b_1} \mathrm{T_n}\, \psi\,(v, u) dz \qquad as\ r \to \infty \quad \ldots\ldots\ldots (1.8)$$

by lemma 2

therefore (1.8) can be written as

$$e^{au} \int_{a_1}^{b_1} \mathrm{T_n} \ \ [\psi\,(z\,,u) - \psi\,(v, u)] dz$$

but by lemma (2) this converges to zero as m→∞ for fixed u combining all the cases we get proof of lemma

Application :-

Consider a solid circular cylinder whose axis is coincident with z axis defined by $-h \le z \le h$ and

$0 \le r \le d$, where d is the radius. Consider the heat conduction problem with symmetry with respect to z axis . i.e ,

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\ \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{k}\ \frac{\partial u}{\partial t} \qquad \ldots\ldots\ldots\ldots\ldots(1.9)$$

where $k = \frac{k}{\rho_0}$ where k is conductibility of medium ρ is calarific capacity assumed to be constant with initial and boundry condition ,

$$u\,(r, h, t) + k_1 \frac{\partial u}{\partial z}\,(r, h, t) = \theta_a = \text{constant} \quad 0 < r < d\ ,\ t > 0$$

$$u\,(r, -h, t) + k_2 \frac{\partial u}{\partial z}\,(r, -h, t) = \theta_b = \text{constant} \quad 0 < r < d\ ,\ t > 0$$

$$u\,(r, z, o) = 0$$

$$u\,(\xi, z, t) = f\,(z, t) \quad -h < z < z\ ,\ t > 0 \qquad o < \xi < d$$

$$u\,(d, z, t) = g\,(z, t) \quad t > 0$$

We shall now have the generalized solution u (r , z ,t) in the space , applying the Laplace Finite Marchi Fasulo transform to equation (1.9) . and taking into account the boundary condition we get

$$\int_0^\infty \int_{-h}^{h} \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\ \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} \right) e^{-pt}\, p_n(z) dt\, dz$$

$$= \int_0^\infty \int_{-h}^{h} \frac{1}{r}\ \frac{\partial u}{\partial t}\ e^{-pt}\ p_n(z) dt\, dz$$

Since $u = u\,(r, z, t)$

$$\frac{\partial^2 U\,(r, n, p)}{\partial r^2} + \frac{1}{r}\ \frac{\partial\, U(r, n, p)}{\partial r} + \int_0^\infty \int_{-h}^{h} \frac{\partial^2 u}{\partial z^2}\ e^{-pt}\, p_n(z)\ dt\, dz$$

$$= \frac{1}{k} \int_0^\infty e^{-pt} \left[\frac{\partial}{\partial t} \int_{-h}^{h} p_n(z)\ u\,(r, z, t) dz \right] dt$$

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{\eta}{p} - a_n{}^2 U = \frac{1}{k}\left[e^{-pt}\int_{-h}^{h} p_n(z) u(r,z,t)\,dz \right]_0^\infty$$

$$-\int_0^\infty -p\, e^{-pt}\int_{-h}^{h} p_n(z)\; u(r,z,t)\,dz\ \Big]\,dt$$

But u (r ,z ,0) =0 and after substituting the limits and by the property that

$$\int_{-h}^{h}\frac{\partial^2 U}{\partial r^2}\, p_n(z)dz = \frac{p_n(h)}{\alpha_1}\left[\beta_1\ u(r,z,t) + \alpha_1\frac{\partial u}{\partial z}\right]_{z=h}$$

$$-\ \frac{p_n(-h)}{\alpha_2}\left[\beta_2\ u(r,z,t) + \alpha_2\frac{\partial u}{\partial z}\right]_{z=-h}$$

and using the condition , substituting $\beta_1 = \beta_2 = 1$, $\alpha_1 = k_1$ $\alpha_2 = k_2$

$$\eta = \left[\frac{p_n(h)\,\theta_a}{k_1} - \frac{p_n(-h)\,\theta_b}{k_2}\right] = \text{constant .}$$

Hence , we get .

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{\eta}{p} - a_n^2 U = \frac{p}{k}U$$

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} - \left(a_n^2 + \frac{p}{k}\right)U = \frac{-\eta}{p}$$

let $\alpha^2 = \left(a_n^2 + \frac{p}{k}\right)$ then,

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} - \alpha^2 U = \frac{-\eta}{p}\,.$$

The solution of this differential equation is ,

$$U(r, n, p) = A\,I_0(\alpha r) + B\,k_0(\alpha r) + \frac{\eta}{p\alpha^2} \qquad \text{.............(1.10)}$$

where $I_0(\alpha r)$ and $k_0(\alpha r)$ are the modified Bessel functions of first and second kind of order zero respectively.

As r tends to zero , $k_0(\alpha r)$ tends to infinity. But $U(r, n, p)$ remains finite ,hence B=0 .

Then equation becomes

$$U(r, n, p) = A\,I_0(\alpha r) + \frac{\eta}{p\alpha^2} \qquad \text{................................(1.11)}$$

Since $u(\xi, z, t) = f(z, t)$

$\Rightarrow$ U (ξ , n, p) = F (n, p) by putting r = ξ

$$U(r, n, p) = F(n, p) = A\, I_0(\alpha\xi) + \frac{\eta}{p\alpha^2}$$

$$A = \frac{F(n,p) - \frac{\eta}{p\alpha^2}}{I_0(\alpha\xi)}$$, putting this value of A in (1.11)

$$U(r, n, p) = \frac{\left[F(n,p) - \frac{\eta}{p\alpha^2} \right]}{I_0(\alpha\xi)} I_0(\alpha r) + \frac{\eta}{p\alpha^2} \qquad \ldots\ldots\ldots..(1.12)$$

Applying the inverse Laplace finite Marchi Fasulo transform to (1.12), we get

$$u(r, z, t) = \lim_{r, m \to \infty} \frac{1}{2\pi i} \sum_{n=1}^{m} \frac{p_n(z)}{\lambda_n} \left\{ \int_{\sigma - ir}^{\sigma + ir} \frac{F(n,p) - \frac{n}{p\alpha^2}}{I_0(\alpha\xi)} I_0(\alpha r) + \frac{\eta}{p\alpha^2} \right\} e^{pt}\, dp .$$

Now for each φ (z ,t) ∈ D (I) it can be shown that

< u(r , z , t), φ (z , t)

$$= \frac{1}{2\pi i} \int_{-h}^{h} \int_{0}^{\infty} \sum_{n=1}^{\infty} \frac{p_n(z)}{\lambda_n} \int_{\sigma - i\infty}^{\sigma + i\infty} \left\{ \left\{ \frac{\left[F(n,p) - \frac{\eta}{p\alpha^2} \right]}{I_0(\alpha\xi)} I_0(\alpha r) + \frac{\eta}{p\alpha^2} \right\} \varphi(z, t)\, dp\, dt\, dz \right\} .$$

So that u(r ,z , t) as a conventional function is obtained as

$$u(r, z, t) = \frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{p_n(z)}{\lambda_n} \int_{\sigma - i\infty}^{\sigma + i\infty} \left\{ \left\{ \frac{\left[F(n,p) - \frac{\eta}{p\alpha^2} \right]}{I_0(\alpha\xi)} I_0(\alpha r) + \frac{\eta}{p\alpha^2} \right\} dp \right\} \ldots \ldots (1.13)$$

We show that (1.13) is the solution since F(n ,p) is bounded on the domain [A , B]× [a_1 , b_1].

it can also be shown that the series obtained by applying D_t , D_z^2, D_u^2 seperatly under the summation and under integral sign converges uniformly on domain D.

To verify the boundry condition we shall show that for each φ ∈ D (I) u ∈ £'$_{+, a}$ (w)

< u(r ,z , t) φ (z ,t) > → < f(z , t) φ (z ,t) >

$$\int_{-h}^{h} \int_{o}^{\infty} \frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{p_n(z)}{\lambda_n} \int_{\sigma - i\infty}^{\sigma + i\infty}$$

$$\left\{ \frac{\left[F(n,p) - \frac{\eta}{p\alpha^2} \right]}{I_0(\alpha\xi)} I_0(\alpha\xi) + \frac{\eta}{p\alpha^2} \right\} dp \quad \varphi(z, t)\, dt\, dz$$

$$= \int_{-h}^{h} \int_{o}^{\infty} \frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{p_n(z)}{\lambda_n} \int_{\sigma - i\infty}^{\sigma + i\infty} F(n, p)\ \varphi(z, t)\, dt\, dz$$

$= \langle f(z,t), \varphi(z,t) \rangle$

$\langle u(\xi, z, t), \varphi(z,t) \rangle =$

Similarly we can show that $\langle u(d,t,z), \varphi(z,t) \rangle = \langle g(z,t), \varphi(z,t) \rangle$

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